

A class of superintegrable systems of Calogero type

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Abstract

We show that the three body Calogero model with inverse square potentials can be interpreted as a maximally superintegrable and multiseparable system in Euclidean three-space. As such it is a special case of a family of systems involving one arbitrary function of one variable.

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1 Introduction

The purpose of this article is to investigate the relation between the rational three-body Calogero model in one-dimension [3] and superintegrable systems in two and three dimensions [5, 7, 16].

The original (quantum) Calogero model was written in the form

$$\left\{ - \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) + \frac{1}{8} \omega^2 [(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2] \right. \\ \left. + \frac{g_1}{(x_2 - x_3)^2} + \frac{g_2}{(x_1 - x_3)^2} + \frac{g_3}{(x_1 - x_2)^2} \right\} \Psi = E \Psi. \quad (1.1)$$

Upon introducing the centre-of-mass coordinate R and the Jacobi relative coordinates ρ and λ [13]

$$R = \frac{1}{3}(x_1 + x_2 + x_3), \quad \rho = \frac{1}{\sqrt{2}}(x_1 - x_2), \quad \lambda = \frac{1}{\sqrt{6}}(x_1 - x_2 - 2x_3) \quad (1.2)$$

equation (1.1) was rewritten [3] as follows

$$\left\{ - \left(\frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial \lambda^2} \right) + \frac{3}{8} \omega^2 (\rho^2 + \lambda^2) + \frac{1}{2} \frac{g_1}{(\sqrt{3}\lambda - \rho)^2} + \frac{1}{2} \frac{g_2}{(\sqrt{3}\lambda + \rho)^2} + \frac{1}{2} \frac{g_3}{\rho^2} \right\} \Psi = E \Psi, \quad (1.3)$$

where the motion of the centre-of-mass has been factored out.

A superintegrable system is one that admits more integrals of motion than it has degrees of freedom. Systematic searches for superintegrable systems of the form

$$H(\mathbf{x}, \mathbf{p}) = \frac{1}{2} \mathbf{p}^2 + V(\mathbf{x}) \quad (1.4)$$

have been conducted in Euclidean spaces \mathbb{E}^n for $n = 2$ and 3 [5, 7, 16]. The classical or quantum Hamiltonian (1.4) is said to be *superintegrable* if it admits $n + k$, $1 \leq k \leq n - 1$ integrals of motion, n of them in involution. It is *minimally superintegrable* for $k = 1$ and *maximally superintegrable* for $k = n - 1$. For $n = 2$ the two cases coincide and superintegrability simply means the existence of three functionally independent integrals of motion (including the Hamiltonian). For $n = 3$ a superintegrable system can have either 4 or 5 functionally independent integrals of motion.

The N body Calogero model [4] (and, in particular, the three body one [3]) is known to be superintegrable [1, 2, 12, 20, 21, 26]. An extensive literature exists on superintegrability in classical and quantum systems of the form (1.4) (see [14, 22, 24] and the references therein) devoted mainly, though not exclusively [8, 9] to systems with integrals of motion of at most second order in the momenta. Superintegrable systems with complete sets of commuting quadratic integrals of motion are *multiseparable*. This means that the corresponding Hamilton-Jacobi, or Schrödinger equation allows the separation of variables in more than one system of (orthogonal) coordinates. Alternatively, multiseparability can be described in terms of the geometric properties of the Killing two-tensors determined by the first integrals of motion that are quadratic in the momenta (see [12] as well as the relevant references therein).

In what follows, we shall deal with the quantum mechanical problem, but all conclusions are the same (*mutatis mutandis*) for the classical ones. For the systems admitting integrals of motion of order three or higher, this is not necessarily the case [8, 9, 11].

2 The Calogero model in the classification of superintegrable systems

In a recent article [12] the *invariant theory of Killing tensors* (see also [17, 18, 25] and the relevant references therein) was used to classify orthogonally separable Hamiltonian systems in the Euclidean space \mathbb{E}^3 . In particular, it was shown that the inverse square Calogero model with the potential

$$V = \frac{1}{(x_1 - x_2)^2} + \frac{1}{(x_2 - x_3)^2} + \frac{1}{(x_3 - x_1)^2} \quad (2.5)$$

allows the (orthogonal) separation of variables in 5 different coordinate systems, namely spherical, circular cylindrical, rotational parabolic, prolate spheroidal and oblate spheroidal (see also [2, 21]).

In this study [12] the potential (2.5) was viewed as a potential in the Hamiltonian (1.4), corresponding to a single particle in a potential field in \mathbb{E}^3 . The potential (2.5) was shown to allow 5 functionally independent first integrals (including the Hamiltonian). From them it is possible to construct 5

inequivalent pairs of integrals in involution (in addition to the Hamiltonian). Each such pair is determined by two Killing tensors that share the same orthogonal eigenvectors, thus generating an orthogonal separable system of coordinates. For example, the spherical coordinate system is generated by the following pencil of Killing tensors (including the metric) whose components given in terms of the Cartesian coordinates (x_1, x_2, x_3) are as follows [12]:

$$\begin{bmatrix} a_1 + c_2 x_3^2 + c_3 x_2^2 & -c_3 x_1 x_2 & -c_2 x_1 x_3 \\ -c_3 x_1 x_2 & a_1 + c_3 x_1^2 + c_2 x_3^2 & -c_2 x_2 x_3 \\ -c_2 x_1 x_3 & -c_2 x_2 x_3 & a_1 + c_2 x_1^2 + c_3 x_2^2 \end{bmatrix}. \quad (2.6)$$

The formula (2.6) can be rewritten as

$$a_1 g^{ij} + c_2 K_1^{ij} + c_3 K_2^{ij}, \quad i, j = 1, 2, 3, \quad (2.7)$$

where K_1^{ij} and K_2^{ij} are the components of two *canonical* Killing tensors \mathbf{K}_1 , \mathbf{K}_2 that share the same orthogonally integrable (i.e., surface forming) eigenvectors and g^{ij} are the components of the metric \mathbf{g} of \mathbb{E}^3 (see [12] for more details).

That notwithstanding, the Calogero potential (2.5) does not appear (at least explicitly) in the list of superintegrable systems in \mathbb{E}^3 , established earlier [5, 16] under the assumption that the first integrals that afford maximal or minimal superintegrability were to be quadratic in the momenta. To unravel this mystery we first observe that the Killing tensors that determine the corresponding integrals of motion obtained for the potential (2.5) in [12] are not in a *canonical* form (as in (2.6), for example), but are rotated with respect to this form. As an example, let us consider again spherical coordinates (r, θ, ϕ) in \mathbb{E}^3 generated by the hypersurfaces of the orthogonally integrable eigenvectors of the Killing tensor (2.6) given by the following coordinate transformations to the Cartesian coordinates (x_1, x_2, x_3) :

$$x_1 = r \sin \theta \cos \phi, \quad x_2 = r \sin \theta \sin \phi, \quad x_3 = r \cos \theta. \quad (2.8)$$

A potential that allows separation in these coordinates must have the form

$$V(r, \theta, \phi) = f(r) + \frac{1}{r^2} g(\theta) + \frac{1}{r^2 \sin^2 \theta} k(\phi) \quad (2.9)$$

and the corresponding additional integrals of motion quadratic in the momenta will be in their standard form, namely

$$\begin{aligned} F_1 &= L_1^2 + L_2^2 + L_3^2 + 2 \left[g(\theta) + \frac{1}{\sin^2 \theta} k(\phi) \right], \\ F_2 &= L_3^2 + 2k(\phi), \end{aligned} \quad (2.10)$$

where $L_i, i = 1, 2, 3$ are the infinitesimal generators of $SO(3)$, that can be determined in terms of the Cartesian coordinates $x_i, i = 1, 2, 3$ as follows: $L_1 = x_2 p_3 - x_3 p_2, L_2 = x_3 p_1 - x_1 p_3, L_3 = x_1 p_2 - x_2 p_1$. Note that the first integrals (2.10) in terms of the Cartesian coordinates can be rewritten as

$$\begin{aligned} F_1 &= K_1^{ij} p_i p_j + U_1(x_1, x_2, x_3), \\ F_2 &= K_2^{ij} p_i p_j + U_2(x_1, x_2, x_3), \end{aligned} \quad (2.11)$$

where $i, j = 1, 2, 3, K_1^{ij}, K_2^{ij}$ are the components of the “spherical” Killing tensors (2.7) and (p_1, p_2, p_3) are the operators $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$ respectively (quantum mechanics case) or the momenta components corresponding to the Cartesian coordinates (x_1, x_2, x_3) (classical mechanics case).

If we rotate the x_1 –, x_2 – and x_3 –axes in (2.8), the form of the potential (2.9) changes, so do the integrals (2.10), but separation of variables will still occur (in spherical coordinates with different axes).

In the case of the potential (2.5) the rotation taking the Killing tensors into their standard form is a non-trivial one, given by [12] (compare with (1.2)):

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 & 0 & \sqrt{2} \\ -1 & \sqrt{3} & \sqrt{2} \\ -1 & -\sqrt{3} & \sqrt{2} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix}. \quad (2.12)$$

Accordingly, for the Calogero potential (2.5) we obtain

$$V = 2 \left[\frac{1}{(\sqrt{3}\tilde{x}_1 - \tilde{x}_2)^2} + \frac{1}{(\sqrt{3}\tilde{x}_1 + \tilde{x}_2)^2} + \frac{1}{\tilde{x}_2^2} \right] \quad (2.13)$$

and we see that the variable \tilde{x}_3 is absent from (2.13). Expressing \tilde{x}_1 and \tilde{x}_2 in terms of spherical coordinates (2.8), we get

$$V = \frac{2}{r^2 \sin^2 \theta} \left[\frac{1}{(\sqrt{3} \cos \phi - \sin \phi)^2} + \frac{1}{(\sqrt{3} \cos \phi + \sin \phi)^2} + \frac{1}{\sin^2 \phi} \right], \quad (2.14)$$

i.e. a potential in the form (2.9) with $f(r) = 0$, $g(\theta) = 0$ and $k(\phi)$ specified.

In what follows we show that after the rotation (2.12) it is possible to see that the Calogero potential (2.13) is a member of an infinite family of potentials, depending on one arbitrary function and sharing a number of important properties, such as superintegrability. Indeed, recall that all superintegrable potentials that separate in spherical coordinates plus at least one other system were derived in [16]. The potential

$$V = \frac{k(\phi)}{r^2 \sin^2 \theta} \quad (2.15)$$

occurs several times. In what follows we list 5 *functionally independent* first integrals (including the Hamiltonian H) that afford multi-separability for the potential (2.15):

$$\begin{aligned} H &= \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \frac{k(\phi)}{r^2 \sin^2 \theta}, \\ F_1 &= L_1^2 + L_2^2 + L_3^2 + \frac{2k(\phi)}{\sin^2 \theta}, \\ F_2 &= L_3^2 + 2k(\phi), \\ F_3 &= \frac{1}{2}p_3^2, \\ F_4 &= L_1 p_2 + p_2 L_1 - p_1 L_2 - L_2 p_1 - 4 \frac{\cos \theta}{r \sin^2 \theta} k(\phi), \end{aligned} \quad (2.16)$$

where $k(\phi)$ is an arbitrary function. The functional independence of the first integrals (2.16) has been verified with the aid of a computer algebra package (i.e., the Jacobian $\frac{\partial(H, F_1, F_2, F_3, F_4)}{\partial(x_1, x_2, x_3, p_1, p_2, p_3)}$ is of rank 5 at a generic point). It is important to note that the functionally independent first integrals (2.16) are *linearly connected*, which means that they are subject to an additional constraint specified by the following expression in terms of the coordinates $\mathbf{x} = (x_1, x_2, x_3)$:

$$f_0(\mathbf{x})H + f_1(\mathbf{x})F_1 + f_2(\mathbf{x})F_2 + f_3(\mathbf{x})F_3 + f_4(\mathbf{x})F_4 = 0, \quad (2.17)$$

where $f_0(\mathbf{x}) = 2x_3^2$, $f_1(\mathbf{x}) = 1$, $f_2(\mathbf{x}) = -1$, $f_3(\mathbf{x}) = -2(x_1^2 + x_2^2 + x_3^2)$, $f_4 = x_3$. This formula is a consequence of the following “rotational” symmetry, that can be defined in a coordinate-free way. We can write all of the expressions in the formula (2.16) as $F_k = K_k^{ij} p_i p_j + U_k$, where $i, j = 1, 2, 3$. Then the

Killing tensor \mathbf{K}_k with the components K_k^{ij} (including the metric) is subject to the following formula

$$\mathcal{L}_{L_3} \mathbf{K}_k = 0, \quad (2.18)$$

where \mathcal{L} denotes the Lie derivative. We also note that the vector space spanned by the quadratic parts of the first integrals (2.16) are invariant with respect to translations along the x_3 -axis.

It is easy to show now that the potential (2.15) is orthogonally separable with respect to other systems of coordinates as well. Indeed, the pairs of involutive first integrals leading to the orthogonal separation of variables in the Schrödinger equation are $\{F_1, F_2\}$ (spherical), $\{F_2, F_3\}$ (circular cylindrical), $\{F_2, F_4\}$ (rotational parabolic), and $\{F_2, F_1 \mp a^2 2F_3\}$ (oblate and prolate spheroidal). Another way to see this is by looking at the separable potentials derived in [16]. In terms of Cartesian coordinates the potential (2.15) is given by:

$$V = \frac{k(x_2/x_1)}{x_1^2 + x_2^2}. \quad (2.19)$$

Recall [16] that the separable potentials corresponding to “rotational” coordinates, namely spherical, circular cylindrical, rotational parabolic, oblate and prolate spheroidal in the Cartesian coordinates (x_1, x_2, x_3) all have the form

$$V = f + g + \frac{k(x_2/x_1)}{x_1^2 + x_2^2}, \quad (2.20)$$

where k are arbitrary functions, while f and g are specified differently in each case. The common part of the five separable potentials is exactly the potential (2.19).

These observations put in evidence that the potential (2.19) defines a family of maximally superintegrable potentials separable with respect to the five “rotational” orthogonal coordinate systems, namely spherical, circular cylindrical, rotational parabolic, oblate and prolate spheroidal whose Killing tensors are constrained by the rotational symmetry condition (2.18). As for the Calogero potential (2.13), in the coordinates $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ determined by the transformation (2.14), it assumes the form (2.19) for

$$k(t) = 2(1 + t^2) \left[\frac{3 + t^2}{(3 - t^2)^2} + 1 \right], \quad (2.21)$$

where $t = \tilde{x}_2/\tilde{x}_1$.

The potential (2.15) can be imbedded into more general families of potentials in \mathbb{E}^3 that are *minimally* superintegrable. In contrast to maximally superintegrable potentials they admit three additional integrals rather than four. They are

$$\begin{aligned} V_1 &= \alpha(x_1^2 + x_2^2 + x_3^2) + \frac{\beta}{x_3^2} + \frac{1}{x_1^2 + x_2^2} h(\phi), \\ V_2 &= \frac{\alpha}{r} + \beta \frac{\cos \theta}{r^2 \sin^2 \theta} + \frac{1}{r^2 \sin^2 \theta} h(\phi), \\ V_3 &= k(x_1^2 + x_2^2) + 4kx_3^2 + \frac{1}{x_1^2 + x_2^2} h(\phi). \end{aligned} \quad (2.22)$$

The potential V_1 with $(\alpha, \beta) \neq (0, 0)$ separates in all of the 5 “rotational” coordinate systems considered above except rotational parabolic ones. V_2 separates only in spherical and rotational parabolic, while V_3 - in cylindrical and rotational parabolic. We mention that a special case of V_2 with $\beta = 0$ and $h(\phi) = \text{const}$ is the Hartmann potential used in molecular physics to describe ring-shaped molecules [10, 15].

The rotation (2.12) in \mathbb{E}^3 has a simple meaning for three particles on a line with inverse square potentials. Comparing (1.3) with (2.14), we see that the rotation corresponds to introducing centre-of-mass coordinates (1.2). If we factor out the centre-of-mass motion (i.e. drop the term $\frac{1}{2}p_3^2$ in the kinetic energy), we reobtain the Hamiltonian (1.3) with $\omega = 0$.

The system (1.3) can be viewed as one particle in a potential in the Euclidean plane \mathbb{E}^2 . Interestingly, it is not multiseparable. For both $\omega = 0$ and $\omega \neq 0$ it separates only in polar coordinates, so it allows only one second order integral of motion (in addition to the Hamiltonian), namely

$$F = L_3^2 - \left[\frac{g_1}{(\sqrt{3} \sin \phi - \cos \phi)^2} + \frac{g_2}{(\sqrt{3} \sin \phi + \cos \phi)^2} + \frac{g_3}{\cos^2 \phi} \right]. \quad (2.23)$$

If the system (1.3) is superintegrable in \mathbb{E}^2 , the second integral of motion must be of higher order in the momenta, not commuting with F given by (2.23). Multiseparability of a physical system, in particular the Calogero model, may also be of interest from the point of view of different possible quantizations. In a recent article Féher *et al* [6] have used separation of variables in circular cylindrical coordinates in the three-body Calogero model to investigate all possible self-adjoint extensions of the corresponding angular and radial Hamiltonians. The question arises whether separation of variables in other coordinates might not lead to different quantizations.

3 Conclusions

The beauty of the Calogero model is lost when its potential is written in the form (2.13). The formula (2.13) does however show that this system is a member of a family of maximally superintegrable systems determined by the general formula (2.15), involving an arbitrary function of one variable, the azimuthal angle ϕ . All of them allow the orthogonal separation of variables in the 5 different “rotational” coordinate systems. The complete set of commuting operators (first integrals) in each case consists of the Hamiltonian H and F_2 of (2.16) and one more operator (F_1, F_3, F_4 and $F_1 \mp a^2 p_3^2$, respectively). The operator F_2 that is thus singled out corresponds, in the case of the free motion, to a one-dimensional subgroup of the (orientation-preserving) isometry group $I(\mathbb{E}^3)$, which is the symmetry group of the Schrödinger equation without a potential. This subgroup generates the angle ϕ , common to all 5 “rotational” orthogonally separable coordinate systems.

This raises the question whether other maximally superintegrable systems involving arbitrary functions exist. All superintegrable systems in \mathbb{E}^3 separating in spherical coordinates and in one further system were found in [16]. All further systems separable in (at least) two coordinate systems were found in [5]. In the lists provided by Evans [5] five systems are maximally superintegrable and each one depends on arbitrary constants. In addition, eight systems are listed as minimally superintegrable, each depending on one arbitrary function and up to three constants. One of the minimally superintegrable systems has the potential

$$V_1 = F(r) + \frac{c_1}{x_1^2} + \frac{c_2}{x_2^2} + \frac{c_3}{x_3^2}, \quad (3.24)$$

where c_1, c_2 and c_3 are arbitrary constants. Here and below r, θ and ϕ are spherical coordinates as specified by (2.8). Its superintegrability is due to the fact that the corresponding Hamiltonian commutes with the operators

$$\begin{aligned} F_1 &= L_1^2 + \frac{2c_2 \cos^2 \theta}{\sin^2 \theta \sin^2 \phi} + \frac{2c_3 \sin^2 \theta \sin^2 \phi}{\cos^2 \theta}, \\ F_2 &= L_2^2 + \frac{2c_1 \cos^2 \theta}{\sin^2 \theta \cos^2 \phi} + \frac{2c_3 \sin^2 \theta \cos^2 \phi}{\cos^2 \theta}, \\ F_3 &= L_3^2 + \frac{2c_1}{\cos^2 \phi} + \frac{2c_2}{\sin^2 \phi}. \end{aligned} \quad (3.25)$$

This potential becomes maximally superintegrable for $F = \omega(x_1^2 + x_2^2 + x_3^2)$. For $c_1 = c_2 = c_3 = 0$ it simply becomes rotationally invariant (but not

maximally superintegrable). Four of the minimally superintegrable potentials have the form

$$V_i(x_1, x_2, x_3) = \tilde{V}_i(x_1, x_2) + f(x_3), \quad i = 2, 3, 4, 5, \quad (3.26)$$

where $\tilde{V}_i(x, y)$ is one of the four multiseparable potentials in \mathbb{E}^2 [7]. In each case the set of integrals of motion consists of

$$F_1 = \frac{1}{2}p_3^2 + f(x_3) \quad (3.27)$$

and three further operators, the principal parts of which lie in the enveloping algebra of the Lie algebra of the isometry group $I(\mathbb{E}^2)$. In particular, for $\tilde{V}_i(x_1, x_2) = 0$ the Hamiltonian and F_1 of (3.27) commutes with the Lie algebra $\{L_3, p_1, p_2\}$, i.e. H and F_1 are invariant under the orientation-preserving isometry group $I(\mathbb{E}^2)$. This provides a total of 4 integrals of motion, never 5. Out of these 4 functionally independent integrals of motion we can form 4 inequivalent triplets of integrals of motion in involution, namely (H, F_1, X_i) , $i = 1, 2, 3, 4$ with

$$X_1 = p_1^2, \quad X_2 = L_3^2, \quad X_3 = L_3 p_1 + p_1 L_3, \quad X_4 = L_3^2 + a^2(p_1^2 - p_2^2),$$

where $a \neq 0$. These triplets correspond to the separation of variables in the cartesian, polar, parabolic translational and elliptic translational coordinates, respectively. Within the $x_1 x_2$ -plane the origin and the orientation of axes can be chosen arbitrarily.

Finally, three of the minimally superintegrable systems depend on an arbitrary function of the azimuthal angle ϕ . They all have the form

$$V_i(r, \theta, \phi) = \tilde{V}_i(r, \theta) + \frac{k(\phi)}{r^2 \sin^2 \theta}, \quad i = 4, 7, 8. \quad (3.28)$$

The integral F_2 of (2.16) is present in each case, together with H and one of F_1, F_3 or F_4 . In particular, for $\tilde{V}_i(r, \theta) = 0$ all of the operators (2.16) are integrals of motion.

We conclude that in \mathbb{E}^3 the potential (2.15) is the only potential that is maximally superintegrable and depends on an arbitrary function (of one variable). The three body Calogero model corresponds to one particular choice of this function, namely that given in (2.15) and (2.21).

An important question arises in this context. Namely, what are the physical consequences in classical and quantum mechanics, of the existence of

a maximally superintegrable system, depending on an arbitrary function? In classical mechanics maximally superintegrable systems have the property that their finite trajectories are closed [19]. In quantum mechanics they have degenerate energy levels and it has been conjectured [23, 22] that they are exactly solvable. We cannot expect these properties to hold for the potential (2.15) with $k(\phi)$ arbitrary. We suspect that the reason for this paradox is that the 5 integrals (2.16) are functionally independent, but linearly connected.

One of the messages that we arrive at is that results considered to be “canonical” in one approach to a problem may be quite non-obvious in another. Thus, the Killing tensors obtained in [12] were not in canonical (standard) form for the Calogero model viewed as an \mathbb{E}^3 problem. The advantage of the invariant approach used in [12, 17, 18, 25] is the following. For a given isometry group action in a vector space of Killing tensors one can employ the approach developed in [12, 17, 18, 25] to determine which orbit a Killing tensor belongs to and then find the corresponding isometry group action mapping the Killing tensor in question to its canonical form (i.e., the corresponding *moving frames map*).

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